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RESEARCH REPORT No. BR-9

A Convergent Asymptotic Representation for Integrals

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CONTRACT No. AF-18(600)-367
DECEMBER 1954

NEW YORK UNIVERSITY
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A CONVERGENT ASYMPTOTIC REPRESENTATION FOR INTEGRALS

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The research reported in this document has been made possible through support and sponsorship extended by the Office of Scientific Research, H. Q., Air Research and Development Command, U. S. A. F., Baltimore, Maryland, under Contract No. AF-18(600)-367. It is published for technical information only, and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1954

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Abstract

This report presents a new method for obtaining an asymptotic representation for integrals of the form $\int_0^\infty e^{-px} x^{c-1} f(x)\, dx$ when p is large. It is shown that if f(x) satisfies certain conditions this representation is also convergent. Numerical calculations seem to show that the first term of the representation gives a close approximation to the value of the integral for a wide range of values of p.

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1. Introduction

In many applications it is important to find representations for the integral

$$\int_0^\infty e^{-px} x^{c-1} f(x)\,dx$$

when the parameter p is large. Here c is a complex constant such that Re c > 0 and f(x) is usually an analytic function. The most commonly used representation, whose justification is due to Watson[1], is obtained by expanding f(x) in a power series in the neighborhood of x = 0 and then integrating term by term. For the sake of convenience, we shall call the representation so obtained Watson's representation, even though it has been known from much earlier times. This representation has several disadvantages. First, in many cases the series obtained is divergent. For example, if we take $f(x) = (1 + x)^{-1}$, then the divergent series

$$\Gamma(c)p^{-c} \sum_0^\infty (-)^n c(c+1)\ldots(c+n-1)p^{-n}$$

is obtained. Second, the representation always has a singularity at p = 0 even though the function defined by the integral may be a regular function of p, as is the case when $f(x) = (1 + x)^{-1}$ and 0 < Re c < 1. Third, the representation is practically useless for computation when p is not very large but only in the range from 2 to 5.

In this paper we present a method for expanding the above integral into a series which, in many cases, is both asymptotic in p and convergent for p > 0; consequently, the series may be used for computation for all non-negative values of p. Moreover, for moderate values of p the first term alone usually gives more accurate results than the asymptotic representation obtained by Watson's method. A further advantage of the first term is that it is not singular for $p = 0$ if $f(x)$ goes to zero like x^{-1} when x goes to infinity. However, it should be remarked that the series obtained in this paper is much more complicated than that obtained by Watson's method and that in many cases the complete series can be obtained only with great difficulty.

2. The First Term Approximation

Let us denote the integral

$$\int_0^\infty e^{-px} \, x^{c-1} \, f(x)dx$$

by the symbol $L_c[f(x)]$. If we wish to make explicit the fact that this integral is a function of p, we shall use the symbol $L_c[f(x); p]$.

As is well known, the first term of Watson's representation for $L_c[f(x)]$ is $f(0)\Gamma(c)p^{-c}$. The error incurred in using this term for the integral $L_c[f(x)]$ is

$$E = \int_0^\infty e^{-px} \, x^{c-1}[f(x)-f(0)]dx = \int_0^\infty e^{-px}x^c \, dx \int_0^1 f'(xv) \, dv$$

if we assume f(x) is continuously differentiable. Suppose that

$$|f'(x)| < M$$

for $0 \leq x < \infty$; then

$$|E| < M\Gamma(c + 1) \, p^{-c-1} \quad .$$

This shows that the ratio of the error to the first term in Watson's representation is of the order p^{-1}.

A better approximation to the value of the integral $L_c[f(x)]$ will be given by the expression

$$(1) \qquad f(\tfrac{c}{p})\,\Gamma(c)\,p^{-c}.$$

Now, the error is

$$E_1 = \int_0^\infty e^{-px}\,x^{c-1}\left\{\frac{f(x) - f(\frac{c}{p})}{x - \frac{c}{p}}\right\}(x - \tfrac{c}{p})\,dx.$$

An integration by parts shows that

$$(2) \qquad E_1 = p^{-1}\int_0^\infty e^{-px}\,x^c\,\frac{d}{dx}\left\{\frac{f(x) - f(\frac{c}{p})}{x - \frac{c}{p}}\right\}dx$$

$$= p^{-1}\int_0^\infty e^{-px}\,x^c\,dx\int_0^1 vf''\left[\frac{c+v}{p} + v(x - \frac{c+1}{p})\right]dv,$$

where we have assumed that $f(x)$ has a continuous second derivative. If we assume also that

$$|f''(x)| < M_1$$

for $0 \leq x < \infty$, then we get

$$(3) \qquad |E_1| < M_1\,\Gamma(c+1)\,p^{-c-2} \quad .$$

This estimate shows that now the ratio of the error to the first term is of the order p^{-2}; consequently, we expect that (1) will give a closer approximation to the value of $L_c[f(x)]$ than the first term in Watson's representation. The following illustrations show the accuracy of our approximation

1) Suppose that $f(x) = (1 + x)^{-1}$ and $c = 1$; then from (1) we get

$$I_1 = \int_0^\infty e^{-px}(1 + x)^{-1}\,dx \sim (1 + p)^{-1}.$$

In the following table we compare for real and imaginary values of p the value of the integral, which we denote by I_1, with the value of the approximation:

p	1	2	3	4	5
I_1	.5963	.3613	.2621	.2063	.1704
app.	.5000	.3333	.2500	.2000	.1667

p	i	2i	3i	4i	5i
I_1	.343-.622i	.212-.399i	.079-.292i	.050-.229i	.034-.188i
app.	.500-.500i	.200-.400i	.100-.300i	.059-.235i	.039-.192i

2) Consider the case where $f(x) = (1+x)^{-1/2}$ and $c = 1$. We get

$$I_2 = \int_0^\infty e^{-px} (1+x)^{-1/2} \, dx \sim (p^2+p)^{-1/2} \quad .$$

In the following table we compare the value of this integral, which we denote by I_2, with the corresponding value of the approximation :

p	4	5	4i	5i
I_2	.2263	.1842	.0268-.2414i	.0179-.1952i
app.	.2236	.1826	.0301-.2444i	.0195-.1971i

3) As a final illustration, we shall obtain an approximation for the Hankel function. We have[2]

$$H_0^{(1)}(y) = \frac{2}{\pi i} \int_0^\infty e^{iy \cosh \eta} \, d\eta \quad .$$

Put $\cosh \eta = 1 + x$; then

$$H_0^{(1)}(y) = \frac{2e^{iy}}{\pi i} L$$

where

$$L = \int_0^\infty x^{-1/2} e^{iyx} (2+x)^{-1/2} dx.$$

If we put $c = 1/2$, $p = -iy$, $f(x) = (2 + x)^{-1/2}$ then the integral L is exactly the type we have been considering; consequently, from (1),

$$\begin{aligned} L &\sim \Gamma(\tfrac{1}{2})(-iy)^{-1/2} \left(2- \frac{1}{2iy}\right)^{-1/2} \\ &= \left(\frac{2\pi}{1-4iy}\right)^{1/2} \\ &= \left(\frac{\pi}{2}\right)^{1/2} e^{i\pi/4} y^{-1/2}\left(1 + \frac{1}{16y^2}\right)^{-1/4} \exp\left[-i \arctan(4y)^{-1}\right] \end{aligned}$$

We see then that

$$H_0^{(1)}(y) \sim \left(\frac{2}{\pi y}\right)^{1/2} e^{i(y - \pi/4)} F$$

where

$$F = \left(1 + \frac{1}{16y^2}\right)^{-1/4} \exp\left[-i \arctan(4y)^{-1}\right] .$$

The usual asymptotic expression for $H_0^{(1)}(y)$ is obtained by taking the limit of F as y approaches ∞, that is, $F = 1$.

The following table shows the closeness of the approximation when F is not replaced by 1.

y	0.6	2	4	5
$H_0^{(1)}(y)$	.912-.309i	.224+.510i	-.397-.017i	-.178-.309i
app.	.918-.370i	.228+.514i	-.398-.017i	-.178-.309i

3. The Complete Expansion of $L_c[f(x)]$

It follows from Equation (2), which gives the error in using one term for $L_c[f(x)]$, that this integral can be written as

$$L_c[f(x)] = f(\tfrac{c}{p})\,\Gamma(c)\,p^{-c} + p^{-1}\,L_{c+1}[f_1(x)] \tag{4}$$

where

$$f_1(x) = \frac{d}{dx}\left\{\frac{f(x)-f(\frac{c}{p})}{x-\frac{c}{p}}\right\}. \tag{5}$$

It is clear that in (4) we may replace c by c + 1 and f(x) by $f_1(x)$ to get

$$L_{c+1}[f_1(x)] = f_1(\tfrac{c+1}{p})\,\Gamma(c+1)\,p^{-c-1} + p^{-1}\,L_{c+2}[f_2(x)] \tag{6}$$

where

$$f_2(x) = \frac{d}{dx}\left\{\frac{f_1(x)-f_1(\frac{c+1}{p})}{x-\frac{c+1}{p}}\right\}.$$

Combining (4) and (6) and continuing this process, we find that

$$L_c[f(x)] = p^{-c}\,\Gamma(c)\sum_{k=0}^{n-1} c(c+1)\ldots(c+k-1)\,p^{-2k}\,f_k(\tfrac{c+k}{p}) + p^{-n}L_{c+n}[f_n(x)] \tag{7}$$

where

$$f_0(x) = f(x), \quad f_{k+1}(x) = \frac{d}{dx}\left\{\frac{f_k(x)-f_k(\frac{c+k}{p})}{x-\frac{c+k}{p}}\right\}. \tag{8}$$

If we let n go to infinity in (7), we obtain the infinite series

$$p^{-c}\,\Gamma(c)\sum_{k=0}^{\infty} c(c+1)\ldots(c+k-1)\,p^{-2k}f_h(\tfrac{c+k}{p}). \tag{9}$$

This series is asymptotic in p in a sense which is made precise by the following theorem:

Theorem I:

Put $c = \alpha + i\beta$, where α is real and positive and β is real. For $\beta \neq 0$, suppose that $f(x)$ is analytic for $\mathrm{Re}(x) > 0$ and that $f, f', \ldots, f^{(2N)}$ satisfy the inequalities

$$(10) \quad |f^{(h)}(x)| \leq M e^{\mu |x|}, \qquad h=0,1,\ldots,2N$$

where M and μ are non-negative constants. Then for $n = 1,2,\ldots, N$, we have as $p \to \infty$

$$(11) \quad L_c[f(x)] = p^{-c}\,\Gamma(c) \sum_{k=0}^{N-1} c(c+1)\ldots(c+k-1)p^{-2k} f_k\left(\frac{c+k}{p}\right) + O(p^{-2N-\alpha}).$$

If $\beta = 0$, that is, if c is real and positive, the assumption that $f(x)$ is analytic for $\mathrm{Re}\ (x) > 0$ may be replaced by the assumption that $f(x)$ is 2N times continuously differentiable for $x > 0$, and then (11) holds with α replaced by c.

In many cases the series (9) is not only asymptotic but also convergent. We shall prove

Theorem II:

Suppose $f(z) = f(x+iy)$ can be represented in the form

$$f(z) = \int_0^\infty e^{-zt}\, d\Psi(t), \qquad x > 0,$$

where $\Psi(t)$ is a complex-valued function which is of bounded variation in each finite interval $0 \leq t \leq T$ and which satisfies the inequality

$$|\Psi(t)| \leq M, \qquad t \geq 0$$

Then for $p > 0$ and $\mathrm{Re}\ c > 0$, the series (9) converges to $L_c[f(x)]$.

This theorem, when applied to the integrals used as illustrations in the previous sections, shows that the series converge.

Proof of Theorem I. Equation (7) shows that we need prove only

Equation (7) shows that we need prove only

$$L_{c+N}\left[f_N(x)\right] = O(p^{-N-\alpha}). \tag{12}$$

First, we prove by induction that

$$|f_k^{(h)}(x)| \leq M_k\, e^{\mu|x|} \tag{13}$$

for h = 0, 1, 2,..., 2N-2k and k = 0, 1, 2,...,N. Here

$$M_k = M \exp\left[p^{-1}\left\{k^2 + k(|c|-1)\right\}\right] .$$

From (8) we have

$$\begin{aligned} f_{k+1}(x) &= \frac{d}{dx}\int_0^1 f_k'\left[p^{-1}(c+k)(1-v) + vx\right] dv \\ &= \int_0^1 v\, f_k''\left[p^{-1}(c+k)(1-v) + vx\right] dv . \end{aligned} \tag{14}$$

Suppose now that (13) holds for some value of k; then differentiating (14) k times with respect to x and using (13) we find that

$$\begin{aligned} \left|f_{k+1}^{(h)}(x)\right| &\leq M_k \int_0^1 v^{k+1} \exp\left[\mu|x| + p^{-1}(|c|+k)\right] dv \\ &\leq M_{k+1}\, e^{\mu|x|} \end{aligned}$$

for h = 0, 1, 2,..., 2N-2k-2. By hypothesis, this inequality holds for k = 0, consequently it holds for $0 \leq k \leq N$.

Now, using (13), we see that

$$\left|L_{c+N}\left[f_N(x)\right]\right| = \left|\int_0^\infty e^{-px}\, x^{c+N-1}\, f_N(x)\, dx\right|$$

$$\leq \int_0^\infty e^{-px}\, x^{N+\alpha-1}\, M_N\, e^{\mu x}\, dx$$

$$= O\left(p^{-N-\alpha}\right),$$

which proves the theorem.

Proof of Theorem II. Before proving this theorem, we shall obtain a more convenient representation for $f_k(\frac{c+k}{p})$. Suppose that $f_k(x) = e^{-\alpha x}$; then from (14) we have

$$f_{k+1}(x) = \int_0^1 v\alpha^2 \exp\left[-\frac{\alpha(c+k)}{p} - \alpha v\left(x - \frac{c+k}{p}\right)\right] dv$$

or, if we put $W = \alpha v$,

$$f_{k+1}(x) = \int_0^\alpha W \exp\left[-(c+k)(\alpha-W)p^{-1}\right] e^{-Wx}\, dW. \tag{15}$$

We can now show that if $f_o = e^{-tx}$, then

$$f_n(x) = \int_0^t \int_0^{W_1} \cdots \int_0^{W_{n-1}} W_1 W_2 \cdots W_n\, A_n\, e^{-W_n x}\, dW_n\, dW_{n-1} \cdots dW_1 \tag{16}$$

for $n = 1, 2, \ldots$, where

$$A_n = \exp\left[-ctp^{-1} - cp^{-1}\sum_1^{n-1} W_k + (c+n-1)p^{-1}W_n\right].$$

The proof is by induction. Formula (15) shows that (16) holds for $n = 1$. Suppose that (16) holds for $n = 1, 2, \ldots k$; then, using (15) with $\alpha = W_n$ and interchanging the order of integration (this is justified if $x > 0$), we get

$$f_{k+1}(x) = \int_0^t \int_0^{W_1} \cdots \int_0^{W_{k-1}} W_1 \cdots W_k \, A_k dW_k \cdots dW_1$$

$$\cdot \int_0^{W_k} W \exp\left[- (c+k)(W_k - W)p^{-1}\right] e^{-Wx} dW.$$

Clearly,

$$A_k \exp\left[-(c+k)(W_k - W)\, p^{-1}\right] = A_{k+1} \, .$$

This shows that (16) is valid for $n = k + 1$ and thus for all values of n.

By the hypothesis of Theorem II,

$$f(x) = \int_0^\infty e^{-tx} \, d\Psi(t).$$

To obtain $f_n(x)$, we note first that we may write

$$f_n(x) = P_n\left[f(x)\right]$$

where P_n is a linear operator acting on the space of 2n-times differentiable functions. In fact, the operator P_n may be defined for $n = 1,2,\ldots,$ recursively from (14) as follows:

$$P_n\left[f(x)\right] = \int_0^1 \frac{d^2}{dx^2} P_{n-1}\left[f\left\{p^{-1}(c+n-1)(1+v)+vx\right\}\right] \frac{dv}{v} \, ,$$

and

$$P_o\left[f(x)\right] = f(x).$$

If $x > 0$, it is easy to justify the fact that

$$f_n(x) = P_n\left[\int_0^\infty e^{-tx} d\Psi(t)\right] = \int_0^\infty d\Psi(t) \, P_n\left[e^{-tx}\right] \, .$$

Using formula (16), we get

$$(17) \qquad f_n(x) = \int_0^\infty d\Psi(t) \int_0^t \int_0^{W_1} \cdots \int_0^{W_{n-1}} W_1 \cdots W_n A_n e^{-W_n x} dW_n \cdots dW_1.$$

Put $x = (c + n)p^{-1}$; then we get

$$(18) \qquad f_n\left(\frac{c+n}{p}\right) = \int_0^\infty d\Psi(t) \, e^{-ct/p} \, I(t),$$

where

$$(19) \qquad I(t) = \int_0^t \int_0^{W_1} \cdots \int_0^{W_{n-1}} W_1 \cdots W_n \exp\left(-\frac{1}{p}\sum_1^n W_k\right) dW_n \cdots dW_1 .$$

The region of integration in (19) is an n-dimensional pyramid with vertex at the origin. Since the integrand is a symmetric function of its n arguments, the value of the integral over the n-dimensional cube $0 \leq W_k \leq t$, $(k = 1,2\ldots,n)$ will be exactly $n!$ times the value of $I(t)$. This shows that

$$I(t) = \frac{1}{n!}\left(\int_0^t W e^{-W/p}\, dW\right)^n$$

$$= \frac{1}{n!}\; p^{2n} \left\{1 - \left(1 + \frac{t}{p}\right) e^{-t/p}\right\}^n .$$

Substituting this value for $I(t)$ in (18), we obtain the desired result:

$$(20) \qquad f_n\left(\frac{c+n}{p}\right) = \frac{1}{n!}\, p^{2n} \int_0^\infty d\Psi(t)\; e^{-ct/p} \left\{1 - \left(1 + \frac{t}{p}\right) e^{-t/p}\right\}^n .$$

With the help of this representation the series (9) may be written as

$$(21) \qquad \sum_{k=0}^\infty \binom{c + k - 1}{k} \int_0^\infty d\Psi(t)\; e^{-ct/p} \left[\phi\left(\frac{t}{p}\right)\right]^k ,$$

where

$$(22) \qquad \phi(t) = 1 - (1 + t)\, e^{-t} .$$

To prove Theorem II, we must now prove that this series is convergent and that its sum has the value

$$(23) \qquad L_c[f(x)] = \int_0^\infty x^{c-1} e^{-px} f(x)dx = \int_0^\infty x^{c-1} e^{-px} dx \int_0^\infty e^{-xt} d\Psi(t)$$

$$= \int_0^\infty d\Psi(t)\, \left(1 + \frac{t}{p}\right)^{-c}$$

The function $\phi(t)$ increases steadily from zero to unity as t increases from 0 to ∞. From the binomial theorem it follows that

$$(24)\qquad (1+\frac{t}{p})^{-c} = e^{-ct/p}\left\{1-\phi(\frac{t}{p})\right\}^{-c} = e^{-ct/p}\sum_{k=0}^{\infty}\binom{c+k-1}{k}\phi^{k}(\frac{t}{p})$$

for $0 \leq t < \infty$. Here we have used the fact that

$$(-)^{k}\binom{-c}{k} = \binom{c+k-1}{k}, \quad k = 0, 1, \ldots \qquad .$$

Inserting (24) into (23) and integrating the first n terms, we find for the integral in (23) the representation

$$(25)\qquad \sum_{0}^{n-1}\binom{c+k-1}{k}\int_{0}^{\infty} e^{-ct/p}\,\phi^{k}(\frac{t}{p})d\Psi(t) + R_n$$

where

$$(26)\qquad R_n = \int_{0}^{\infty} e^{-ct/p}\sum_{n}^{\infty}\binom{c+k-1}{k}\phi^{k}(\frac{t}{p})d\Psi(t).$$

From (21) and (25) it follows that the theorem will be proved when it has been shown that $R_n \to 0$ as $n \to \infty$.

For the sake of convenience, put

$$\omega(t) = \Psi(tp);$$

then $\omega(t)$ is of the bounded variation in each finite t-interval, and $|\omega(t)| \leq M$. Replacing t by tp in (26), we get

$$(27)\qquad R_n = \int_{0}^{\infty} e^{-ct}\sum_{n}^{\infty}\binom{c+k-1}{k}\phi^{k}(t)\,d\omega(t).$$

We will now prepare to integrate by parts. The series in the integrand may be differentiated term by term; the derivative of the integrand is, therefore, equal to

$$(28)\qquad e^{-ct}\left\{-c\sum_{k=n}^{\infty}\binom{c+k-1}{k}\phi^{k}(t)+\sum_{k=n}^{\infty}k\binom{c+k-1}{k}\phi^{k}(t)\,te^{-t}\right\}.$$

Using the identity

$$te^{-t}=\frac{t}{1+t}\left\{1-\phi(t)\right\},$$

we may write the second summation in (28) in the form

$$(29)\qquad \frac{t}{1+t}\sum_{k=n}^{\infty}k\binom{c+k-1}{k}\left\{\phi^{k-1}(t)-\phi^{k}(t)\right\}.$$

After partial summation, (29) becomes

$$(30)\qquad \begin{aligned}&\frac{t}{1+t}\cdot n\binom{c+n-1}{n}\phi^{n-1}(t)+\frac{t}{1+t}\sum_{k=n}^{\infty}\left[(k+1)\binom{c+k}{k+1}-k\binom{c+k-1}{k}\right]\phi^{k}(t)\\ &=\frac{t}{1+t}\cdot n\binom{c+n-1}{n}\phi^{n-1}(t)+\frac{ct}{1+t}\sum_{k=n}^{\infty}\binom{c+k-1}{k}\phi^{k}(t).\end{aligned}$$

Here we have used the identities

$$(k+1)\binom{c+k}{k+1}-k\binom{c+k-1}{k}=c\binom{c+k-1}{k}.$$

Using (30), we may now write the derivative (28) in the form

$$(31)\qquad \frac{te^{-ct}}{1+t}\cdot n\binom{c+n-1}{n}\phi^{n-1}(t)-\frac{ce^{-ct}}{1+t}\sum_{k=n}^{\infty}\binom{c+k-1}{k}\phi^{k}(t).$$

We are now ready to integrate by parts in (27). There is no doubt that we can do this, since $\omega(t)$ is bounded and since the integrand can be written as the sum of a finite number of terms in the form

$$(32) \qquad (1+t)^{-c} - e^{-ct} \sum_{k=0}^{n-1} \binom{c+k-1}{k} \phi^k(t) \, .$$

Observe, incidentally, that for $n \geq 1$ the integrand (32) vanishes at $t = 0$ and $t = \infty$. For $n \geq 1$ we have, therefore,

$$R_n = c \int_0^\infty \frac{\omega(t)e^{-ct}}{1+t} \sum_{k=n}^{\infty} \binom{c+k-1}{k} \phi^k(t)dt - n\binom{c+n-1}{n} \int_0^\infty \frac{\omega(t)te^{-ct}}{1+t} \phi^{n-1}(t)dt.$$

Therefore,

$$(33) \quad |R_n| \leq |c|M \int_0^\infty \frac{e^{-\alpha t}}{1+t} \sum_{k=n}^{\infty} \left|\binom{c+k-1}{k}\right| \phi^k(t)dt + M_n \left|\binom{c+n-1}{n}\right| \int_0^\infty e^{-\alpha t}\phi^{n-1}(t)dt.$$

Since the terms in the infinite series are greater than or equal to 0, we may invert the order of summation and integration in (33) to obtain

$$(34) \qquad |R_n| \leq M|c|S_n + MT_n,$$

where

$$(35) \qquad S_n = \sum_{k=n}^{\infty} \left|\binom{c+k-1}{k}\right| \int_0^\infty \frac{e^{-\alpha t}\phi^k(t)}{1+t} \, dt,$$

$$(36) \qquad T_n = \left|\binom{c+n-1}{n}\right| \int_0^\infty e^{-\alpha t}\phi^{n-1}(t) \, dt \, .$$

To complete the proof of the theorem, it is sufficient to show that $S_n \to 0$ and $T_n \to 0$ as $n \to \infty$.

First consider S_n. Our task is to show that S_n is the remainder, $\sum_{k=n}^{\infty}$, of a convergent series. For $k > 2\alpha$, we write

$$(37) \qquad \int_0^{\infty} e^{-\alpha t} \phi^k(t) \frac{dt}{1+t} = \int_0^{\log \frac{k}{2\alpha}} + \int_{\log \frac{k}{2\alpha}}^{\infty} .$$

We have, for $k > 2\alpha$,

$$\phi^k(\log \tfrac{k}{2\alpha}) = \exp\left\{ k \log \phi (\log \tfrac{k}{2\alpha}) \right\}$$

$$(38) \qquad = \exp\left\{ k \log \left[1 - (1 + \log \tfrac{k}{2\alpha}) \tfrac{2\alpha}{k}\right] \right\}$$

$$< \exp\left\{ - (1 + \log \tfrac{k}{2\alpha}) \cdot 2\alpha \right\} = \alpha A k^{-2\alpha},$$

where αA is a finite constant depending upon α, but not upon k. By the steadily-increasing property of $\phi(t)$, the first integral on the right-hand side (37) is, therefore, less than

$$(39) \qquad \phi^k(\log \tfrac{k}{2\alpha}) \int_0^{\infty} e^{-\alpha t} \, dt < A k^{-2\alpha} .$$

For the second integral on the right-hand side in (37) we have the upper bound

$$\int_{\log \frac{k}{2\alpha}}^{\infty} e^{-\alpha t} \left\{ 1 - (1 + \log \tfrac{k}{2\alpha}) e^{-t} \right\}^k \frac{dt}{1+t}$$

$$< \frac{1}{\log \frac{k}{2\alpha}} \int_{\log \frac{k}{2\alpha}}^{\infty} e^{-\alpha t} \left\{ 1 - (1 + \log \tfrac{k}{2\alpha}) e^{-t} \right\}^k dt .$$

After the change of variable $\tau = (1 + \log \frac{k}{2\alpha}) e^{-t}$, we see that the last expression is less than

$$(40) \qquad (\log \tfrac{k}{2\alpha})^{-1-\alpha} \int_0^1 \tau^{\alpha-1} (1-\tau)^k d\tau = \frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)} (\log \tfrac{k}{2\alpha})^{-1-\alpha} .$$

Using the estimates (39), (40) for the integrals in (37), we find, from the definition (35),

$$(41) \qquad S_n \leq A \sum_{k=n}^{\infty} \left| \binom{c+k-1}{k} \right| k^{-2a} + \sum_{k=n}^{\infty} \left| \binom{c+k-1}{k} \right| \frac{\Gamma(a)\Gamma(k+1)}{\Gamma(a+k+1)} \left(\log \frac{k}{2a}\right)^{-1-a} .$$

Since, according to a well-known property of the Γ- function,

$$\binom{c+k-1}{k} = \frac{\Gamma(c+k)}{\Gamma(c)\Gamma(k+1)} \simeq \frac{k^{c-1}}{\Gamma(c)} \quad \text{as} \quad k \to \infty,$$

there exists a constant B, depending upon c but not upon k, such that

$$(42) \qquad \left| \binom{c+k-1}{k} \right| \leq B\, k^{a-1} .$$

Similarly, there is a constant B' such that

$$(43) \qquad \frac{\Gamma(a)\Gamma(k+1)}{\Gamma(a+k+1)} \leq B'\, k^{-a} .$$

Using (42) and (43) in (41), we find

$$S_n \leq AB \sum_{k=n}^{\infty} k^{-1-a} + BB' \sum_{k=n}^{\infty} k^{-1} \left(\log \frac{k}{2a}\right)^{-1-a} .$$

Since both of these series have finite sums, we have

$$(44) \qquad S_n \to 0 \quad \text{as} \quad n \to \infty .$$

Now we have only to show that $T_n \to 0$ as $n \to \infty$. We have, for any number $K > 0$,

$$T_n = n \left| \binom{c+n-1}{n} \right| \left(\int_0^K + \int_K^\infty \right) e^{-at} \phi^{n-1}(t)\, dt$$

$$\leq B\, n^a \left(\phi^{n-1}(K) \cdot \frac{1}{a} + \int_0^\infty e^{-at} \left\{ 1 - (1+K)e^{-t} \right\}^{n-1} dt \right) .$$

Since $\emptyset(K) < 1$, it follows that

$$T_n \leq O(1) + B_n^a \int_K^\infty e^{-at} \left\{ 1 - (1 + K)\, e^{-t} \right\}^{n-1} dt.$$

Setting $\tau = (1 + K)\, e^{-t}$, we find

$$\begin{aligned} T_n &\leq O(1) + B_n^a (1+K)^{-a} \int_0^1 \tau^{a-1} (1-\tau)^{n-1} dt \\ &= O(1) + B_n^a (1+K)^{-a} \frac{\Gamma(a)\,\Gamma(n)}{\Gamma(a+n)} \\ &\leq O(1) + B_n^a (1+K)^{-a} \cdot B'(n-1)^{-a}; \end{aligned}$$

here B and B' have the same meaning as in (42) and (43). Letting $n \to \infty$, we find

$$\limsup_{n \to \infty} T_n \leq BB'(1+K)^{-a} \quad .$$

Since K is an arbitrary positive number, the right-hand side of this inequality may be made arbitrarily small. Therefore,

$$T_n \to 0 \quad \text{as} \quad n \to \infty.$$

This completes the proof of the theorem.

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(CLASSIFICATION)
Security Information

Bibliographical Control Sheet

1. Originating agency and/or monitoring agency:

O.A.: Institute of Mathematical Sciences, Division of Electromagnetic Research, New York University, New York City
M.A.: Mathematics Division, Office of Scientific Research

2. Originating agency and/or monitoring agency report number:

O.A.: Research Report No. BR-9
M.A.: OSR-TN-54-342

3. Title and classification of title:

A Convergent Asymptotic Representation for Integrals (UNCLASSIFIED)

4. Personal author(s): J. Franklin and B. Friedman

5. Date of report: November, 1954

6. Pages: i + 1 - 18

7. Illustrative material: none

8. Prepared for Contract No.(s): AF-18(600)367

9. Prepared for Project Code(s) and/or No.(s): No. R-354-10-15

10. Security classification: (UNCLASSIFIED)

11. Distribution limitations: none

12. Abstract:

This report presents a new method for obtaining an asymptotic representation for integrals of the form $\int_0^\infty e^{-px} x^{c-1} f(x)dx$ when p is large. It is shown that if f(x) satisfies certain conditions this representation is also convergent. Numerical calculations seem to show that the first term of the representation gives a close approximation to the value of the integral for a wide range of values of p.

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